22. Calcolo integrale: esercizi

Esercizio 22.26. *Calcolare*

1. $\displaystyle\int \big(2x - 5x^2 + \sqrt{3x+1}\big)\,dx,$ *2.* $\displaystyle\int \Big(\frac{4}{1+x^2} - x^4 + 5\,\mathrm{e}^{2x}\Big)\,dx,$

3. $\displaystyle\int \Big(\frac{3^x}{12} - \frac{2}{1+x} - 2x^5\Big)\,dx,$ *4.* $\displaystyle\int \Big(\frac{3}{x^5} - 2\cos(5x) + \frac{1}{x-5}\Big)\,dx.$

$\boxed{R}$ *1.* Con l'ausilio delle tabelle e per le proprietà delle primitive, si ha

$$\int \big(2x - 5x^2 + \sqrt{3x+1}\big)\,dx = 2\int x\,dx - 5\int x^2\,dx + \frac{1}{3}\int 3(3x+1)^{1/2}\,dx$$
$$= x^2 - \frac{5}{3}x^3 + \frac{1}{3}\,\frac{(3x+1)^{3/2}}{3/2} + c = x^2 - \frac{5}{3}x^3 + \frac{2}{9}(3x+1)^{3/2} + c.$$

2. Con l'ausilio delle tabelle e per le proprietà delle primitive, si ha

$$\int \Big(\frac{4}{1+x^2} - x^4 + 5\,\mathrm{e}^{2x}\Big)\,dx = 4\int \frac{1}{1+x^2}\,dx - \int x^4\,dx + \frac{5}{2}\int 2\,\mathrm{e}^{2x}\,dx$$
$$= 4\arctan x - \frac{x^5}{5} + \frac{5}{2}\,\mathrm{e}^{2x} + c.$$

3. Con l'ausilio delle tabelle e per le proprietà delle primitive, si ha

$$\int \Big(\frac{3^x}{12} - \frac{2}{1+x} - 2x^5\Big)\,dx = \frac{1}{12}\int 3^x\,dx - 2\int \frac{1}{1+x}\,dx - 2\int x^5\,dx$$
$$= \frac{3^x}{12\ln 3} - 2\ln|1+x| - \frac{x^6}{3} + c.$$

4. Con l'ausilio delle tabelle e per le proprietà delle primitive, si ha

$$\begin{aligned}\int \Big(\frac{3}{x^5} - 2\cos(5x) + \frac{1}{x-5}\Big)\,dx &= \\ &= 3\int x^{-5}\,dx - \frac{2}{5}\int 5\cos(5x)\,dx + \int \frac{1}{x-5}\,dx \\ &= -\frac{3}{4}x^{-4} - \frac{2}{5}\operatorname{sen}(5x) + \ln|x-5| + c.\end{aligned}$$

Esercizio 22.27. *Discutere l'esistenza degli integrali*

1. $\displaystyle\int_0^1 \frac{2x+1}{3x+2}\,dx,$ *2.* $\displaystyle\int_0^1 \Big(x + \frac{3-x}{1+x^2}\Big)\,dx,$

3. $\displaystyle\int_{1/12}^{1/6} \frac{1-6x}{x-3x^2}\,dx,$ *4.* $\displaystyle\int_0^1 \frac{3t+1}{1+t^2}\,dt,$

5. $\displaystyle\int_0^{1/2} \frac{1-5t}{\sqrt{1-t^2}}\,dt,$ *6.* $\displaystyle\int_{-1}^0 \frac{x^2-3}{x^4-9}\,dx,$

e, se esistono, calcolarli.

$\boxed{R}$ *1.* La funzione integranda è continua nell'intervallo $[0,1]$, quindi è integrabile. Cerchiamo di far comparire al numeratore $3x+2$ in modo da scomporre nella somma di due integrali. Essendo

$$\frac{2x+1}{3x+2} = \frac{2}{3}\frac{3x+2-\frac{1}{2}}{3x+2},$$

si ha

$$\begin{aligned}\int_0^1 \frac{2x+1}{3x+2}\,dx &= \frac{2}{3}\int_0^1 \Big(1 - \frac{1}{2}\frac{1}{3x+2}\Big)\,dx \\ &= \frac{2}{3}\Big[x - \frac{1}{6}\log|3x+2|\Big]_0^1 = \frac{2}{3} - \frac{1}{9}\log\frac{5}{2}.\end{aligned}$$

2. La funzione integranda è continua nell'intervallo $[0,1]$, quindi è integrabile. Per la formula fondamentale del calcolo integrale si ha

$$\begin{aligned}\int_0^1 \Big(x + \frac{3-x}{1+x^2}\Big)\,dx &= \int_0^1 x\,dx + 3\int_0^1 \frac{1}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx \\ &= \Big[\frac{x^2}{2}\Big]_0^1 + 3\Big[\arctan x\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,\frac{d(1+x^2)}{dx}\,dx \\ &= \frac{1}{2} + \frac{3\pi}{4} - \frac{1}{2}\Big[\log(1+x^2)\Big]_0^1 = \frac{1}{2} + \frac{3\pi}{4} - \frac{\log 2}{2}.\end{aligned}$$

3. L'integrale esiste e vale $2\log 2 - \log 3$.

4. La funzione integranda è continua nell'intervallo $[0,1]$, quindi è integrabile. L'integrale indefinito è

$$\int \frac{3t+1}{1+t^2}\,dt = \frac{3}{2}\int \frac{2t}{1+t^2}\,dt + \int \frac{1}{1+t^2}\,dt = \frac{3}{2}\log(t^2+1) + \arctan t + c.$$

Per la formula fondamentale del calcolo integrale si ha allora

$$\int_0^1 \frac{3t+1}{1+t^2}\,dt = \Big[\frac{3}{2}\log(t^2+1) + \arctan t\Big]_0^1 = \frac{3}{2}\log 2 + \frac{\pi}{4}.$$

5. La funzione integranda è continua nell'intervallo $[0,1/2]$, quindi è integrabile. Il suo integrale indefinito è

$$\int \frac{1-5t}{\sqrt{1-t^2}}\,dt = \int \frac{1}{\sqrt{1-t^2}}\,dt + 5\int \frac{-2t}{2\sqrt{1-t^2}}\,dt = \arcsin t + 5\sqrt{1-t^2} + c.$$

Per la formula fondamentale del calcolo integrale si ha

$$\int_0^{1/2} \frac{1-5t}{\sqrt{1-t^2}}\,dt = \arcsin t + 5\sqrt{1-t^2}\Big|_0^{1/2} = \frac{\pi}{6} + 5\big(\frac{\sqrt{3}}{2} - 1\big).$$

6. L'integrale esiste e vale $\pi/(6\sqrt{3})$.

Esercizio 22.28. *Usando le prime due righe della tabella* 2 *di pagina 212, calcolare*

1. $\displaystyle\int_0^{\pi/2} \operatorname{sen}^3 x \cos x\,dx$; *2.* $\displaystyle\int_a^b \operatorname{sen}^3 x\,dx$;

3. $\displaystyle\int_0^1 \frac{x^2}{(1+x^3)^{1/2}}\,dx$; *4.* $\displaystyle\int_0^x \frac{2t+2}{t^2+t+1}\,dt$;

5. $\displaystyle\int_a^b \frac{10^x}{10^x+1}\,dx$; *6.* $\displaystyle\int_a^b \operatorname{tg} x\,dx,\ -\frac{\pi}{2} < a \le b < \frac{\pi}{2}$;

7. $\displaystyle\int_0^{\pi/2} \frac{\operatorname{sen} x}{\cos x - 3}\,dx$; *8.* $\displaystyle\int_0^1 \frac{1}{2x^2-2x+1}\,dx$.

$\boxed{R}$ *1.* Poiché l'integranda si presenta nella forma $f(x)^\alpha f'(x)$, con $f(x) = \operatorname{sen} x$ e $\alpha = 3$, usando le tabelle si ha

$$\int_0^{\pi/2} \operatorname{sen}^3 x \cos x\,dx = \Big[\frac{\operatorname{sen}^4 x}{4}\Big]_0^{\pi/2} = \frac{1}{4}.$$

2. Osservato che $\operatorname{sen}^3 x = \operatorname{sen} x \ \operatorname{sen}^2 x = \operatorname{sen} x - \cos^2 x \ \operatorname{sen} x$ si ha

$$\int_a^b \operatorname{sen}^3 x \, dx = \Big[-\cos x + \frac{\cos^3 x}{3} \Big]_a^b.$$

3. Osservato che

$$\frac{d}{dx}(1+x^3)^{1/2} = \frac{3}{2} \frac{x^2}{(1+x^3)^{1/2}}$$

si ha

$$\int_0^1 \frac{x^2}{(1+x^3)^{1/2}} \, dx = \frac{2}{3} \Big[\sqrt{1+x^3} \Big]_0^1 = \frac{2}{3}(\sqrt{2}-1).$$

4. Osservato che $\frac{d}{dt}(t^2+t+1) = 2t+1$, decomponendo in somma si ottiene

$$\int_0^x \frac{2t+2}{t^2+t+1} \, dt = \log(x^2+x+1) + \frac{2}{\sqrt{3}} \Big(\operatorname{arctg} \frac{2x+1}{\sqrt{3}} - \operatorname{arctg} \frac{1}{\sqrt{3}} \Big).$$

5. Osservato che $\frac{d}{dx}(10^x+1) = 10^x \log 10$ e usando la tabella 2 si ha

$$\int_a^b \frac{10^x}{10^x+1} \, dx = \frac{1}{\log 10} \log \Big| \frac{10^b+1}{10^a+1} \Big|.$$

6. Essendo $\operatorname{tg} x = \dfrac{\operatorname{sen} x}{\cos x}$, si ha

$$\int_a^b \operatorname{tg} x \, dx = \log \frac{|\cos a|}{|\cos b|}.$$

7. Poiché la funzione al numeratore è, a meno del segno, derivata di quella al denominatore, usando la tabella 2 si ha

$$\int_0^{\pi/2} \frac{\operatorname{sen} x}{\cos x - 3} \, dx = -\int_0^{\pi/2} \frac{-\operatorname{sen} x}{\cos x - 3} \, dx = -\int_1^0 \frac{1}{t-3} \, dt = \int_0^1 \frac{1}{t-3} \, dt$$
$$= \Big[\log|t-3| \Big]_0^1 = \log \frac{2}{3}.$$

8. L'integrale esiste e vale $\pi/2$.

Esercizio 22.29. *Usando le prime due righe della tabella 2 di pagina 212, calcolare*

1. $\displaystyle\int \Big(\frac{7}{3}x^5 - \cos(2x) + \frac{1-2x}{1+x-x^2}\Big) \, dx$, *2.* $\displaystyle\int \Big(\frac{3x}{\sqrt{1-x^2}} + 5x^3 + \operatorname{sen} x\Big) \, dx$,

3. $\displaystyle\int \Big(\frac{3-2x}{\sqrt{1-x^2}} + 2x^2 - \frac{1}{3\cos^2 x}\Big) \, dx$, *4.* $\displaystyle\int \Big(\frac{1+2x}{1+x^2} - 7x^3 + 2\operatorname{sen} x\Big) \, dx$,

5. $\displaystyle\int \Big(3x^4 - \frac{1+2x}{\sqrt{1-x^2}} + \frac{5}{x}\Big) \, dx$, *6.* $\displaystyle\int \Big(\frac{3\cos x}{5+\operatorname{sen} x} - \frac{x^6}{2}\Big) \, dx$.

$\boxed{R}$ *1.* Si ha

$$\begin{aligned}
&\int \left(\frac{7}{3}x^5 - \cos(2x) + \frac{1-2x}{1+x-x^2}\right) dx \\
&\quad = \frac{7}{3}\int x^5\,dx - \frac{1}{2}\int 2\cos(2x)\,dx + \int \frac{1}{1+x-x^2}(1+x-x^2)'\,dx \\
&\quad = \frac{7}{18}x^6 - \frac{1}{2}\operatorname{sen}(2x) + \ln|1+x-x^2| + c.
\end{aligned}$$

2. Si ha

$$\begin{aligned}
&\int \left(\frac{3x}{\sqrt{1-x^2}} + 5x^3 + \operatorname{sen} x\right) dx = \\
&\quad = -3\int \frac{-2x}{2\sqrt{1-x^2}}\,dx + 5\int x^3\,dx - \int(-\operatorname{sen} x)\,dx \\
&\quad = -3\sqrt{1-x^2} + \frac{5}{4}x^4 - \cos x + c.
\end{aligned}$$

3. Si ha

$$\begin{aligned}
&\int \left(\frac{3-2x}{\sqrt{1-x^2}} + 2x^2 - \frac{1}{3\cos^2 x}\right) dx = \\
&\quad = 3\int \frac{1}{\sqrt{1-x^2}}\,dx + \int (1-x^2)^{-1/2}\cdot\frac{d(1-x^2)}{dx}\,dx + \\
&\qquad\qquad + 2\int x^2\,dx - \frac{1}{3}\int \frac{1}{\cos^2 x}\,dx \\
&\quad = 3\arcsin x + 2\sqrt{1-x^2} + \frac{2}{3}x^3 - \frac{1}{3}\tan x + c.
\end{aligned}$$

4. Si ha

$$\begin{aligned}
&\int \left(\frac{1+2x}{1+x^2} - 7x^3 + 2\operatorname{sen} x\right) dx \\
&= \int \frac{1}{1+x^2}\,dx + \int (1+x^2)^{-1}\cdot\frac{d(1+x^2)}{dx}\,dx - 7\int x^3\,dx + 2\int \operatorname{sen} x\,dx \\
&= \arctan x + \log(1+x^2) - \frac{7}{4}x^4 - 2\cos x + c.
\end{aligned}$$

5. Si ha

$$\int \left(3x^4 - \frac{1+2x}{\sqrt{1-x^2}} + \frac{5}{x}\right) dx$$
$$= 3\int x^4\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx + \int (1-x^2)^{-1/2} \cdot \frac{d(1-x^2)}{dx}\,dx + 5\int \frac{1}{x}\,dx$$
$$= \frac{3}{5}x^5 - \arcsin x + 2(1-x^2)^{1/2} + 5\ln|x| + c.$$

6. Si ha

$$\int \left(\frac{3\cos x}{5+\operatorname{sen} x} - \frac{x^6}{2}\right) dx = 3\int \frac{1}{5+\operatorname{sen} x}\,(5+\operatorname{sen} x)'\,dx - \frac{1}{2}\int x^6\,dx$$
$$= 3\ln(5+\operatorname{sen} x) - \frac{x^7}{14} + c.$$

Esercizio 22.30. *Calcolare*

1. $\displaystyle\int \frac{\cos x + x\operatorname{sen}^2 x}{\operatorname{sen} x}\,dx;$ *2.* $\displaystyle\int (3x^2-2)\cos x\,dx;$

3. $\displaystyle\int (1-3x)\cos(2x)\,dx;$ *4.* $\displaystyle\int \mathrm{e}^{3x}(x^2+1)\,dx;$

5. $\displaystyle\int (x^2-3x)\ln x\,dx;$ *6.* $\displaystyle\int (3z^2-2z+1)\cos z\,dz;$

7. $\displaystyle\int (t^2-t+2)\ln t\,dt;$ *8.* $\displaystyle\int_0^{\pi/2} x(2x-1)\operatorname{sen} x\,dx.$

$\boxed{R}$ *1.* Si ha

$$\int \frac{\cos x + x\operatorname{sen}^2 x}{\operatorname{sen} x}\,dx = \int \frac{\cos x}{\operatorname{sen} x}\,dx + \int x\operatorname{sen} x\,dx.$$

Il primo integrale:

$$\int \frac{\cos x}{\operatorname{sen} x}\,dx = \int \frac{(\operatorname{sen} x)'}{\operatorname{sen} x}\,dx = \log|\operatorname{sen} x| + c$$

Integrando il secondo per parti, si ha

$$\int x\operatorname{sen} x\,dx = x(-\cos x) - \int 1\cdot(-\cos x)\,dx = -x\cos x + \operatorname{sen} x + c.$$

In definitiva si ha

$$\int \frac{\cos x + x \operatorname{sen}^2 x}{\operatorname{sen} x}\, dx = \log|\operatorname{sen} x| - x\cos x + \operatorname{sen} x + c.$$

2. Integrando due volte per parti, si ha

$$\begin{aligned}\int (3x^2-2)\cos x\, dx &= (3x^2-2)\operatorname{sen} x - \int 6x \operatorname{sen} x\, dx \\ &= (3x^2-2)\operatorname{sen} x - \Big(6x(-\cos x) - \int 6(-\cos x)\, dx\Big) \\ &= (3x^2-2)\operatorname{sen} x + 6x\cos x - 6\operatorname{sen} x + c.\end{aligned}$$

3. Integrando per parti, si ha

$$\begin{aligned}\int (1-3x)\cos(2x)\, dx &= (1-3x)\frac{\operatorname{sen}(2x)}{2} - \int (-3)\frac{\operatorname{sen}(2x)}{2}\, dx \\ &= (1-3x)\frac{\operatorname{sen}(2x)}{2} - \frac{3}{4}\int 2(-\operatorname{sen}(2x))\, dx \\ &= (1-3x)\frac{\operatorname{sen}(2x)}{2} - \frac{3}{4}\cos(2x) + c.\end{aligned}$$

4. Integrando per parti, si ha

$$\begin{aligned}\int \mathrm{e}^{3x}(x^2+1)\, dx = &= \frac{\mathrm{e}^{3x}}{3}(x^2+1) - \int \frac{\mathrm{e}^{3x}}{3} 2x\, dx \\ &= \frac{\mathrm{e}^{3x}}{3}(x^2+1) - \frac{2}{3}\Big(\frac{\mathrm{e}^{3x}}{3}x - \int \frac{\mathrm{e}^{3x}}{3}\, dx\Big) \\ &= \frac{\mathrm{e}^{3x}}{3}(x^2+1) - \frac{2}{9}\mathrm{e}^{3x} x + \frac{2}{27}\mathrm{e}^{3x} + c \\ &= \frac{\mathrm{e}^{3x}}{27}(9x^2 - 6x + 11) + c.\end{aligned}$$

5. Integrando per parti, si ha

$$\begin{aligned}\int (x^2-3x)\ln x\, dx &= \big(\frac{x^3}{3} - \frac{3}{2}x^2\big)\ln x - \int \big(\frac{x^3}{3} - \frac{3}{2}x^2\big)\cdot\frac{1}{x}\, dx \\ &= \big(\frac{x^3}{3} - \frac{3}{2}x^2\big)\ln x - \int \big(\frac{x^2}{3} - \frac{3}{2}x\big)\, dx \\ &= \big(\frac{x^3}{3} - \frac{3}{2}x^2\big)\ln x - \big(\frac{x^3}{9} - \frac{3}{4}x^2\big) + c.\end{aligned}$$

6. Si ha

$$\begin{aligned}\int (3z^2 - 2z + 1)\cos z\, dz &= (3z^2 - 2z + 1)\operatorname{sen} z - \int (6z-2)\operatorname{sen} z\, dz \\ &= (3z^2 - 2z + 1)\operatorname{sen} z - \Big((6z-2)(-\cos z) - \int 6(-\cos z)\, dz\Big) \\ &= (3z^2 - 2z + 1)\operatorname{sen} z + (6z-2)\cos z - 6\operatorname{sen} z + c \\ &= (3z^2 - 2z - 5)\operatorname{sen} z + (6z-2)\cos z + c.\end{aligned}$$

7. Si ha

$$\begin{aligned}\int (t^2 - t + 2)\ln t\, dt &= \big(\frac{t^3}{3} - \frac{t^2}{2} + 2t\big)\ln t - \int \big(\frac{t^3}{3} - \frac{t^2}{2} + 2t\big)\frac{1}{t}\, dt \\ &= \big(\frac{t^3}{3} - \frac{t^2}{2} + 2t\big)\ln t - \int \big(\frac{t^2}{3} - \frac{t}{2} + 2\big)\, dt \\ &= \big(\frac{t^3}{3} - \frac{t^2}{2} + 2t\big)\ln t - \frac{t^3}{9} + \frac{t^2}{4} - 2t + c.\end{aligned}$$

8. $2\pi - 5$.

Esercizio 22.31. *Dopo avere verificato che una primitiva della funzione* $(x+1)\,\mathrm{e}^x$ *è* $x\,\mathrm{e}^x$, *calcolare l'integrale*

$$\int_1^2 \big[(x+1)\,\mathrm{e}^x\,\big]\log x\, dx.$$

$\boxed{R}$ Utilizzando la formula d'integrazione per parti si ottiene

$$\begin{aligned}\int_1^2 \big[(x+1)\,\mathrm{e}^x\,\big]\log x\, dx &= \Big[x\,\mathrm{e}^x\log x\Big]_1^2 - \int_1^2 \frac{x\,\mathrm{e}^x}{x}\, dx \\ &= 2\,\mathrm{e}^2\log 2 - \Big[\,\mathrm{e}^x\,\Big]_1^2 = 2\,\mathrm{e}^2\log 2 - \mathrm{e}^2 + \mathrm{e}\,.\end{aligned}$$